TN 2 & 3: Higher order approx.

Recall: height slope

$$T_1(x) = \underbrace{f(b)}_{\text{height}} + \underbrace{f'(b)}_{\text{slope}}(x-b)$$

$$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2}\underbrace{f''(b)}_{\text{concavity}}(x-b)^2$$

Error Bounds

- On interval [a,b], if $|f''(x)| \le M$, then $|f(x) - T_1(x)| \le \frac{M}{2}|x-b|^2$.
- On interval [a,b], if $|f'''(x)| \le M$, then $|f(x) - T_2(x)| \le \frac{M}{6}|x-b|^3$.

$f(x) = 4x^2 - 5x + \ln(x)$

$\hookrightarrow f(1) = -1$

$f'(x) = 8x - 5 + \frac{1}{x} \rightarrow f'(1) = 4$

$f''(x) = 8 - \frac{1}{x^2} \longrightarrow f''(1) = 7$

$f'''(x) = \frac{2}{x^3} \longrightarrow f'''(1) = 2$

Entry Task *(continuing from last class)*

Let $f(x) = 4x^2 - 5x + \ln(x)$.

a) Find the second Taylor polynomial, based at b = 1.

b) Give a bound on the error over the interval $0.75 \le x \le 1.25$.

$$\boxed{T_2(x) = -1 + 4(x-1) + \frac{7}{2}(x-1)^2}$$

① $|f'''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{0.75^3} \rightarrow 4.74$ M

Find a maximum $0.75 \le x \le 1.25$

② $|f(x) - T_2(x)| \le \frac{4.74}{6}|x-1|^3$

③ $|f(x) - T_2(x)| \le \frac{4.74}{6}|0.25|^3$

$$\boxed{|f(x) - T_2(x)| \le 0.01234}$$

$\hookrightarrow$ more accurate than first Taylor polynomial

Example:
Given an error, find the interval
Again, $f(x) = 4x^2 - 5x + \ln(x)$.
Give a value of a so that the error bound on the interval
$$1 - a \le x \le 1 + a$$
is less than or equal to 0.01.

$$\boxed{1 - 0.23706 \le x \le 1 + 0.23706}$$

① $|f'''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{(1-a)^3}$

$1-a \le x \le 1+a$

② $|f(x) - T_2(x)| \le \frac{\frac{2}{(1-a)^3}}{6} |x - 1|^3$

$1+a$

$|f(x) - T_2(x)| \le \frac{\frac{2}{(1-a)^3}}{6} |a|^3$

$|f(x) - T_2(x)| \le \frac{1}{6} \cdot \frac{2}{(1-a)^3} a^3$

$\frac{1}{3} \frac{a^3}{(1-a^3)} = 0.01$

$\frac{a^3}{(1-a^3)} = 0.03$

$\left(\frac{a}{1-a}\right)^3 = 0.03$

$\frac{a}{1-a} = (0.03)^{1/3}$

$a = (0.03)^{1/3} - a(0.03)^{1/3}$

$a + a(0.03)^{1/3} = (0.03)^{1/3}$

$a(1 + (0.03)^{1/3}) = (0.03)^{1/3}$

$a = \frac{(0.03)^{1/3}}{[1 + (0.03)^{1/3}]}$

$$\boxed{a = 0.23706}$$

Taylor Approximation Idea:
If two functions have ***all*** the same derivative values, then they are the same function (up to a constant).
Let's compare derivatives of $f(x)$ and $T_2(x)$ at b.

$$
\begin{aligned}
T_2(x) &= f(b) + f'(b)(x-b) + \tfrac{1}{2} f''(b)(x-b)^2 \\
T_2'(x) &= 0 \quad + f'(b) \quad + \quad f''(b)(x-b) \\
T_2''(x) &= 0 \quad + \; 0 \quad + \; f''(b) \\
T_2'''(x) &= 0
\end{aligned}
$$

$T_2(b) = f(b)$ height
$T_2'(b) = f'(b)$ slope
$T_2''(b) = f''(b)$ concavity

Now plug in x = b to each of these.

- What do you see?
- Why did we need a ½ ? → $\frac{1}{2}(2x) = x$
- What would $T_3(x)$ look like? $+\frac{1}{6} f'''(b)(x-b)^3$
- What would $T_4(x)$ look like? $+\frac{1}{24} f''''(b)(x-b)^4$

$\underbrace{(T_5(x)?, T_6(x)?\ldots)}$

$\frac{1}{\text{factorial}}$ in front

n$^{\text{th}}$ Taylor polynomial

$$f(b)+f'(b)(x-b)+\frac{1}{2}f''(b)(x-b)^2+\frac{1}{3!}f'''(b)(x-b)^3+\cdots+\frac{1}{n!}f^{(n)}(b)(x-b)^n$$

In sigma notation: $\boxed{0! = 1}$

$$T_n(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(b)(x-b)^k$$

Taylor's Inequality (error bound):
on a given interval [a,b],
if $|f^{(n+1)}(x)| \le M$, then

$$|f(x)-T_n(x)| \le \frac{M}{(n+1)!}|x-b|^{n+1}$$

Example: Find the 9$^{\text{th}}$ Taylor polynomial for $f(x) = e^x$ based at $b = 0$, and give an error bound on the interval $[-2,2]$.

$f(x)=e^x$
$f'(x)=e^x$
$f''(x)=e^x$ ...

$$T_9(x) = 1(x-0)+\frac{1}{2!}1(x-0)^2+\frac{1}{3!}1(x-0)^3+\ldots+\frac{1}{9!}1(x-0)^9$$

$$e^x \approx 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\frac{1}{6!}x^6+\frac{1}{7!}x^7+\frac{1}{8!}x^8+\frac{1}{9!}x^9 \pm$$

① $|f^{(10)}(x)| = |e^x| \le e^2$
$-2 \le x \le 2$

② $|f(x)-T_9(x)| \le \dfrac{e^2}{10!}|x-0|^{10}$

③ $\dfrac{e^2}{10!}(2^{10}) = 0.002085$

error

(accurate to 2 digits)

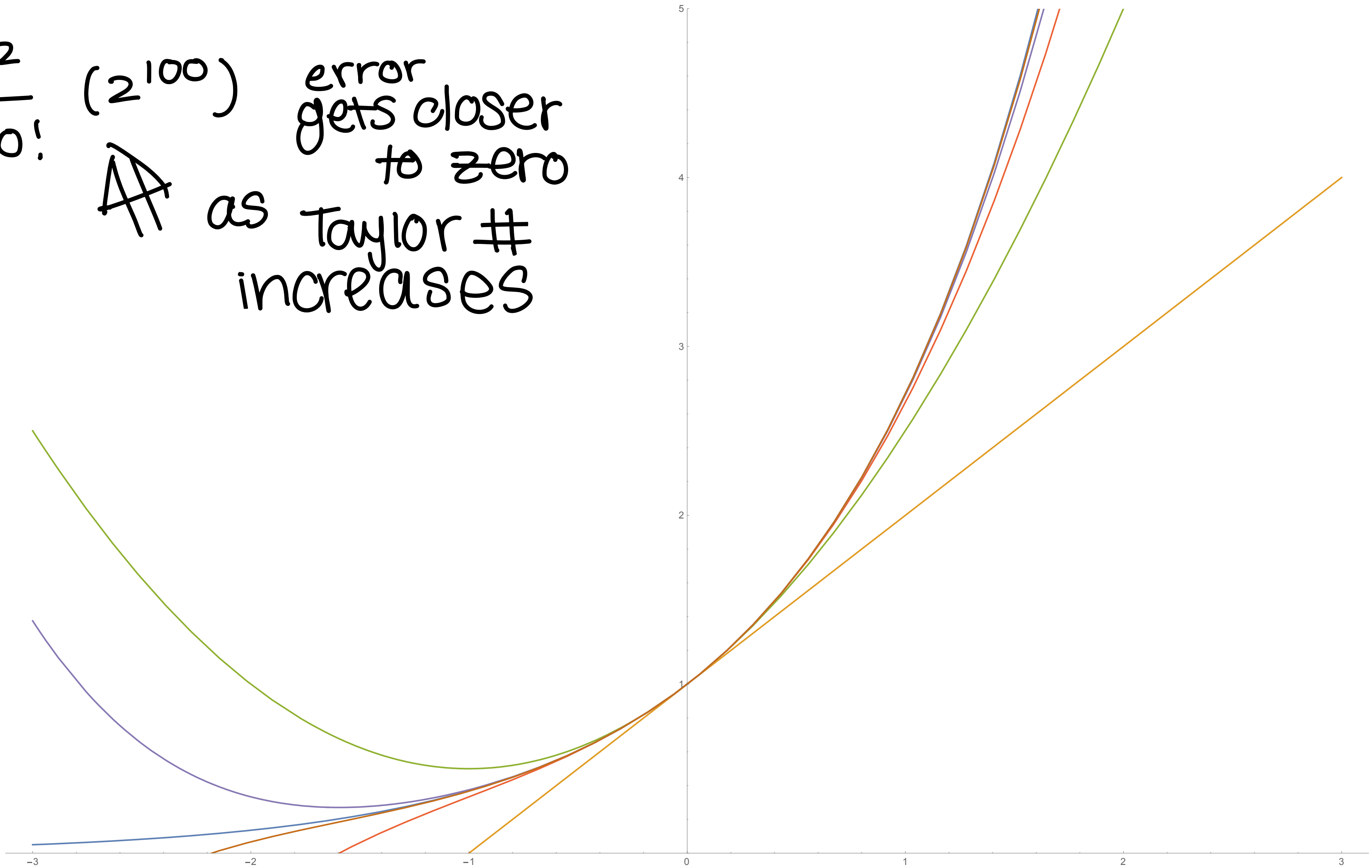
Pattern finding

$f(x) = e^x$ and

$T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$, $T_5(x)$

$\frac{e^2}{100!}(2^{100})$

error gets closer to zero as Taylor # increases



Visuals:
https://www.desmos.com/calculator/cejglhfouh

Example: Again consider,

$$f(x) = e^x \text{ based at } b = 0$$

Find the first value of *n* when Taylor's inequality gives an error less than 0.0001 on $[-2,2]$.

Side Note:

For a fixed constant, a, the expression $\frac{a^k}{k!}$ goes to zero as k goes to infinity.

So the expression $\frac{1}{(n+1)!}|x-b|^{n+1}$, will always go to zero as *n* gets bigger.

Which means that the error goes to zero, unless something unusual is happening with M, which will see in examples later.

Another example:

Find the 8^{th} Taylor Polynomial and error bound on the interval [-1,1] for the following function

$f(x) = \sin(x)$

Visual for Sine: https://www.desmos.com/calculator/titbu0mu5z